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Weibull modulus of fracture strength of toughened ceramics subjected to small-scale contacts

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The Weibull modulus of toughened ceramics was evaluated. The fracture of the toughened ceramics was assumed to be initiated by surface cracks induced during small-scale contacts such as grinding and polishing and an exponential function was selected to describe the *R*-curve behavior. Based on a brief theoretical analysis, a numerical simulation procedure was designed to predict the fracture strength for toughened ceramics with different *R*-curve characteristics. The Weibull modulus of each toughened ceramic was estimated and compared with that of the un-toughened base material. It was concluded that an increase in Weibull modulus can always result from toughening. The increase in Weibull modulus was found to be related directly to the relative crack tolerance, i.e., the ratio of the initial crack size to the critical crack size. This suggests that the improvement in crack stability due to toughening is the main reason for the increased Weibull modulus.

1. Introduction

As a consequence of the natural variability in size, location and severity of the preexisting defects, the measured fracture strength for a given ceramic usually shows a large scatter and should be analyzed by the well-known Weibull distribution function [1, 2],

$$P_{\mathrm{f}} = 1 - \exp\left[-\left(\frac{\sigma_{\mathrm{f}}}{\sigma_0}\right)^m\right] \tag{1}$$

where P_{f} is the cumulative probability of fracture, σ_0 and m are Weibull parameters, the scale parameter and the Weibull modulus, respectively. The Weibull modulus, m, sometimes called the shape parameter, has a value between 5 and 20 for typical ceramics. According to the standard statistics theory, a higher m would lead to a steeper function and thus a lower dispersion of fracture strength.

Recently, existence of *R*-curve behavior has been established for a number of ceramic materials including partially stabilized zirconia (PSZ) [3], coarse grained alumina [4, 5], silicon nitride with elongated grain structure [6, 7] and a series of whisker- or particle-reinforced ceramic composites [8–10]. *R*-curve behavior dictates the functional dependence of crack resistance, K_{R}, on the crack size, c, and has a profound influence on the mechanical properties of ceramics. Especially, it was generally suggested that *R*-curve behavior offers the potential benefits of flaw tolerance and increased Weibull modulus, important factors for increasing reliability of ceramic components. By assuming that the *R*-curve can be described by a simple power-low,

$$K_{\mathrm{R}} = Ac^n \tag{2}$$

(where A and n are empirical constants), several authors [11–13] have shown that the Weibull modulus, m_{T}, of a toughened ceramic can be related to modulus, m_{U}, of the original, or un-toughened brittle ceramic by

$$\frac{m_{\mathrm{T}}}{m_{\mathrm{U}}} = \frac{0.5}{0.5 - n} \tag{3}$$

For many toughened ceramics, the parameter n is usually less than 0.5. Consequently, an increased Weibull modulus, m_{T}, is obtained from Equation 3.

Although a simple analytical formulation, Equation 3, for the Weibull modulus can be deduced, several authors [12, 13] have pointed out that the power law is inconsistent with the partial stability of small surface cracks observed in some toughened ceramics. Consequently, alternative descriptions of *R*-curves should be employed for the analysis of the strength characteristics of toughened ceramics.

The purpose of this paper is to further explore the *R*-curve effect on Weibull modulus of toughened ceramics by selecting an exponential equation to describe the *R*-curve behavior. We concentrated our analysis directly on a general case that the fracture of the considered materials is controlled by small-scale-contact-induced surface cracks. The reason why we conducted

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the study on such a special case is that contact-induced surface cracks have been confirmed to be the worst defects which initiate the fracture of test specimens for a given ceramic in most situation [14–16]. On the other hand, assuming the existence of only one type of fracture-initiating defects avoids the complexity concerning the effect of co-existence of difference flaw populations on the Weibull statistical analysis [17].

2. Theoretical background

In the previous studies, several empirical equations, including a power law [18], an arctan function [19] and an exponential function [20], have been proposed for the description of the measured R-curves for toughened ceramics. We selected the exponential function which was proposed originally by Ramachandran and Shetty [20] for the present study. By noting that there is always a plateau in the measured and/or the theoretically derived R-curve and a lower-bound fracture resistance is physically realistic, Ramachandran and Shetty chose the following function to fit the measured R-curve

$$K_{\mathrm{R}} = K_{\mathrm{H}} - (K_{\mathrm{H}} - K_{\mathrm{L}})\exp\left(-\frac{c}{\lambda}\right) \tag{4}$$

where K_{H}, K_{L} and λ are adjustable parameters. According to the analysis of Ramachandran and Shetty [20], K_{H} and K_{L} may be considered to be estimate of the toughness of the toughened ceramic and that of the original un-toughened ceramic, respectively. On the other hand, λ approximately reflects the range of crack extension over which toughening effects should develop and saturate. In other words, λ may be used as a rough measure for the size of the steady-state process zone formed behind the crack-tip during crack extension.

Several studies [6, 21–23] have confirmed that Equation 4 may give good fits to the R-curve data measured by directly observing the stable growth of indentation-induced cracks located on the surface of the test specimens. Furthermore, Ramachandran and Shetty [21] showed that the fracture strength of indented specimen can be accurately predicted by a combined numerical and graphical procedure determining the point of tangency between crack-driving forces and the R-curves described by Equation 4. These previous studies suggest that Equation 4 is suitable for describing the R-curve behavior of toughened ceramics.

As reviewed by Evans [14], contact-induced cracks form whenever a hard particle plastically penetrates the surface, as inevitably occurs during grinding, polishing and any other abrasive surface-finishing process. The cracks develop in response to a residual stress that results from the creation of a confined plastic zone in the vicinity of the contact site. In other words, during the contact event, the only driving force for crack initiation and propagation is the residual stress. For the sake of convenience, in the present study, we assumed that all the surface cracks formed during contact events can be equivalently treated to be half-penny in shape and the residual stress may be modeled as being concentrated at a point located at the crack center at the elastic-plastic interface. Thus, the driving force, K_{r}, for the evolution of the contact-induced surface crack may be characterized as [24, 25]

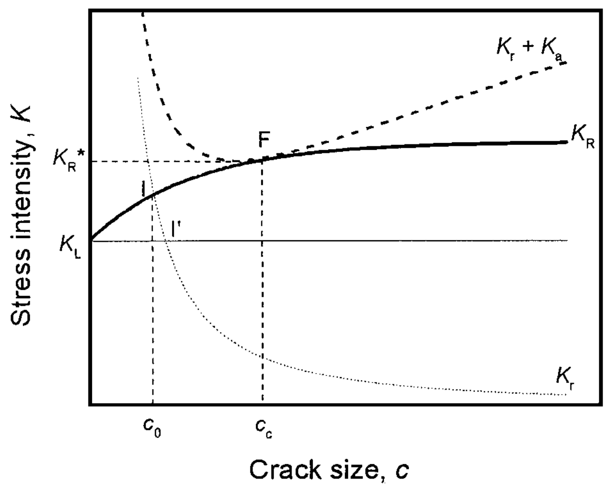

Figure 1 Typical stress intensity factors as functions of crack size.

$$K_{\mathrm{r}} = \frac{\chi F}{c^{3/2}} \tag{5}$$

where c is the radius of the half-penny crack, F is the peak contact load, and χ is a numerical constant dependent on the contact geometry and the elastic/plastic properties of the test material. We refer to the term χF as the equivalent contact load in the following context.

Fig. 1 compares the $K_{\mathrm{r}}(c)$ curve with the $K_{\mathrm{R}}(c)$ curve. As can be seen, $\mathrm{d}K_{\mathrm{r}}/\mathrm{d}c < \mathrm{d}K_{\mathrm{R}}/\mathrm{d}c$ for all c. According to fracture mechanics theory, during the contact event, the crack would propagate stably until $K_{\mathrm{r}} = K_{\mathrm{R}}$ and then an equilibrium position will be attained at the point I where both curves intersect with each other. Thus the equilibrium crack size, c_0, i.e., the initial size of the crack in the specimen subjected to contact, may be obtained by equating Equations 4 and 5, i.e.,

$$\frac{\chi F}{c_0^{3/2}} = K_{\mathrm{H}} - (K_{\mathrm{L}} - K_{\mathrm{L}})\exp\left(-\frac{c_0}{\lambda}\right) \tag{6}$$

The horizontal line in Fig. 1 represents the fracture resistance curve for the un-toughened base material whose fracture toughness is given by K_{L}. For the base material, the initial crack size, $(c_0)_{\mathrm{B}}$ can be given by [13]

$$(c_0)_{\mathrm{B}} = \left(\frac{\chi F}{K_{\mathrm{L}}}\right)^{2/3} \tag{7}$$

When the specimen containing contact-induced surface crack is subjected to an applied stress, σ_{a}, the total stress intensity at the crack tip is given by [24, 25]

$$K = K_{\mathrm{a}} + K_{\mathrm{r}} \tag{8}$$

where the applied stress intensity K_{a} takes the general form for half-penny shaped surface cracks [13]

$$K_{\mathrm{a}} = 2\Omega\sigma_{\mathrm{a}}\left(\frac{c}{\pi}\right)^{1/2} \tag{9}$$

where Ω is a free-surface correction factor for stress intensity. For small half-penny cracks relative to strength test specimen dimension, Ω takes a value of 1.21 at the surface intersection point of the cracks [26].

According to the standard fracture mechanics theory [27], incremental crack extension can occur when K is equal to or greater than the fracture resistance, K_R and an equilibrium position will be attained at $K = K_R$ if $\mathrm{d}K/\mathrm{d}c < \mathrm{d}K_R/\mathrm{d}c$. The criterion for the onset of crack-extension instability at $K = K_R$, leading finally to rupture, is the point of common tangency, point F in Fig. 1. Thus, the fracture strength of the specimen can be obtained by substituting Equations 4 and 8 into the following equations [27],

$$\begin{cases} K = K_R \\ \dfrac{\mathrm{d}K}{\mathrm{d}c} = \dfrac{\mathrm{d}K_R}{\mathrm{d}c} \end{cases} \tag{10}$$

For the un-toughened base material, $K_R = K_L$. Thus the fracture strength, $(\sigma_f)_B$ of the specimen can be deduced directly from Equation 10 and has the form [13]

$$(\sigma_f)_B = \frac{3\pi^{1/2}K_L}{8\Omega(c_c)_B^{1/2}} \tag{11}$$

where $(c_c)_B$ is the critical size crack, i.e., crack size corresponding to the instability point, which is given by

$$(c_c)_B = \left(\frac{4\chi F}{K_L}\right)^{2/3} = 2.5(c_0)_B \tag{12}$$

Since an explicit expression for the fracture strength cannot be deduced from Equation 9, a numerical simulation procedure was designed based on the above analyses in the present study to investigate the effect of R-curve behavior on the strength distribution of toughened ceramics.

3. Numerical simulation procedure

Previous studies have shown that, for a given material, the fracture strengths of a batch of specimens which were ground under the same condition exhibit a significant variation and follow a Weibull distribution described as Equation 1 [1, 2, 14]. Such a variation in strength is primarily related to the spectrum of machining force, which dictate the range of crack sizes, c_0, generated in the surface [14]. Suppose we have 100 ground specimens of an un-toughened base material. The fracture strengths of the 100 specimens are controlled by the contact-induced surface cracks and follow a Weibull distribution with a set prescribed values of m_U and σ_0. Thus, the fracture strength of each specimen, σ_i, may be calculated by substituting a computer-generated random number between 0 and 1 for the fracture probability, P_f, in the alternative form of Equation 1,

$$\sigma_i = \sigma_0\left[\ln\left(\frac{1}{1-P_f}\right)\right]^{1/m_U} \tag{13}$$

Then the critical crack length, $(c_c)_B$, for each specimen was claculated directly from Equation 11 using the resultant σ_i and a prescribed K_L-value. Substituting the resultant 100 $(c_c)_B$-values into Equation 12, respectively, 100 (χF) data were yielded and these data can be used as a *sample* of equivalent contact load.

Now we imagine the above-mentioned 100 specimens were toughened to possess rising R-curve behavior and the toughness characteristics of the toughened specimens can be described with Equation 4 with a set of prescribed values of K_H, K_L and λ. The 100 toughened specimens were ground under the same condition used for the above-mentioned un-toughened base material. That is, the i-th toughened specimen was subjected to a contact event with an equivalent contact load $(\chi F)_i$, the i-th data appeared in the χF sample obtained above. Following the analysis conducted in the preceding section, the initial crack size, c_0, the critical crack size, c_c, and the fracture strength, σ_f, for each specimen were then determined with Equations 6 and 10, respectively, using the Newton-Rhapson iteration method.

The resultant 100 strength data for the toughened specimens were then analyzed to estimate the Weibull modulus with the simple method proposed by Gong and Li [28]. To do so, the coefficient of variation, c_{var}, of the 100 strength data was calculated and then the Weibull modulus was estimated directly with the following Equation [28],

$$m_T = \frac{1.250}{c_{var}} - 0.562 \tag{14}$$

To gain a basic understanding of the R-curve behavior on the Weibull modulus, m_T, of toughened ceramics, the above-mentioned simulation procedure with toughened specimens was repeated by fixing $\sigma_0 = 500$ MPa and $K_L = 5$ MPa$\cdot$m$^{1/2}$ and systematically adjusting the prescribed values of the parameters m_U, K_H and λ.

4. Results and discussion

Rewriting Equation 1 in the form of natural logarithm, we have

$$\ln\ln\left(\frac{1}{1-P_f}\right) = m\ln\sigma_f - m\ln\sigma_0 \tag{15}$$

Equation 15 shows that a straight line with a slope of m in the $\ln\ln 1/(1-P_f)$ versus $\ln\sigma_f$ plot would be obtained if the fracture strength follows the Weibull distribution. In Fig. 2, the strength data yielded from the numerical simulation procedure for two "imagined" toughened ceramics are plotted against the fracture probability in a $\ln\ln 1/(1-P_f)$ versus $\ln\sigma_f$ scale, respectively, and compared with those of the corresponding un-toughened base materials. In these plots, the fracture probability, $(P_f)_i$, for the i-th strength data was calculated with the commonly used Equation [1]

$$(P_f)_i = \frac{i-0.5}{N} \tag{16}$$

where $N = 100$ is the total number of the strength data.

As can be seen, the linearity between $\ln\ln 1/(1-P_f)$ and $\ln\sigma_f$ is evident for each toughened ceramic, implying that the strength variability of the toughened

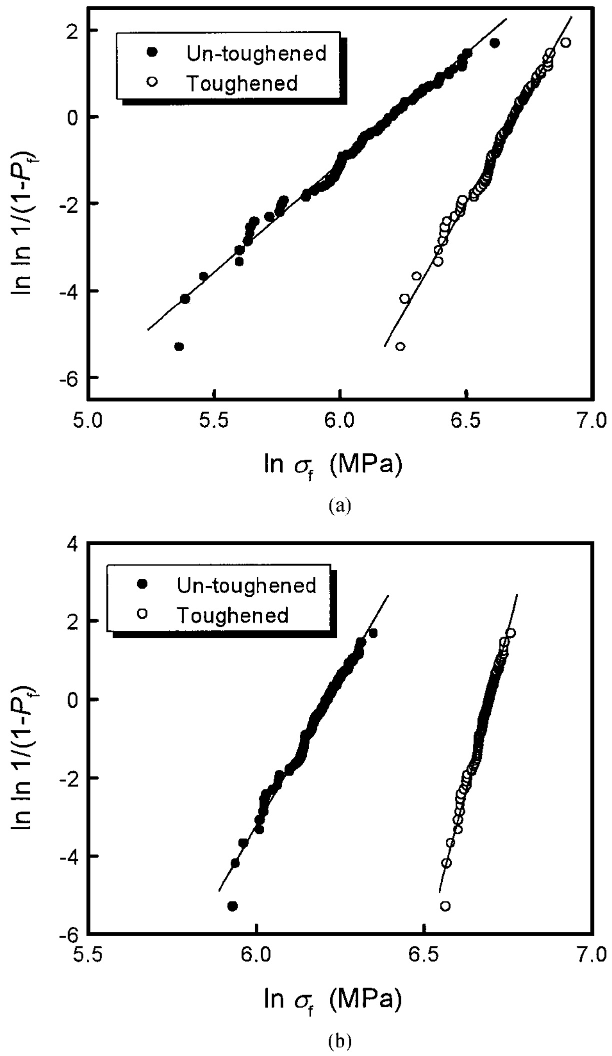

Figure 2 Comparison between the strength distributions of toughened and un-toughened base materials. (a) Un-toughened base material has a Weibull modulus $m_U = 5$; the R-curve behavior of the toughened material is described with $K_H = 10$ MPa·m$^{1/2}$ and $\lambda = 50$ μm. (b) Un-toughened base material has a Weibull modulus $m_U = 15$; the R-curve behavior of the toughened material is described with $K_H = 10$ MPa·m$^{1/2}$ and $\lambda = 50$ μm.

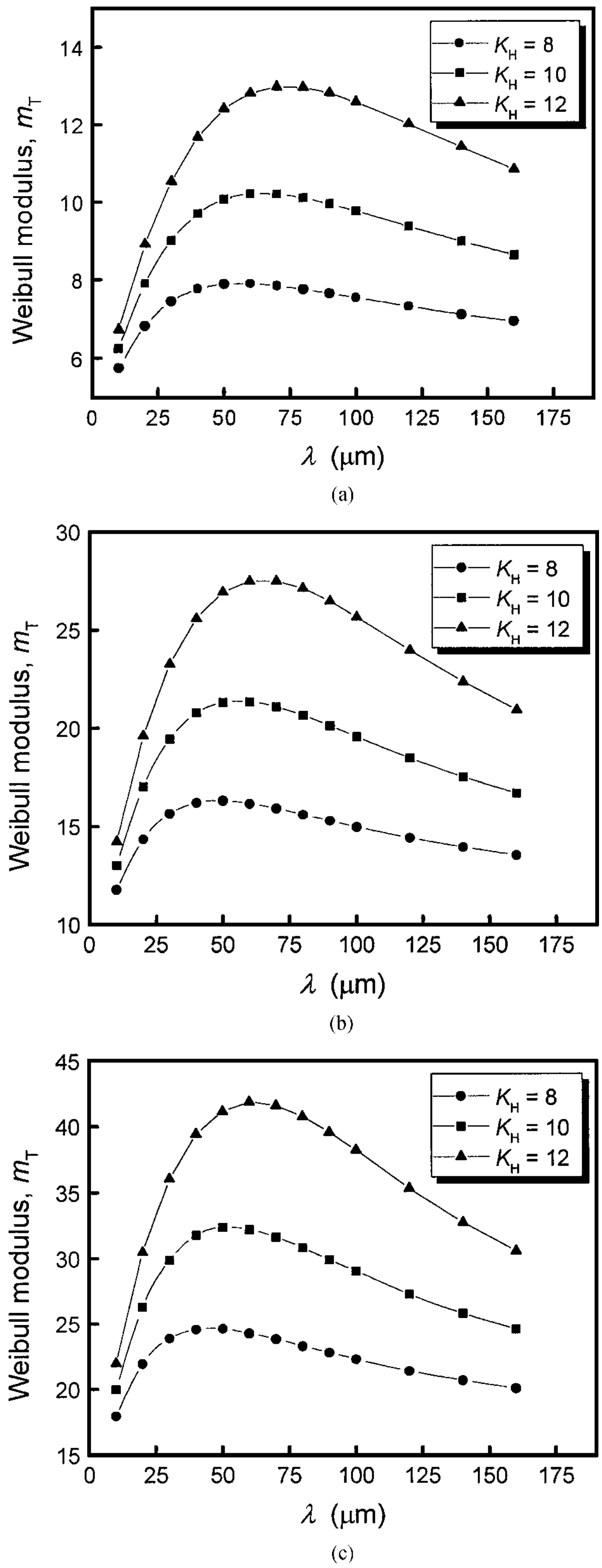

Figure 3 Variation of Weibull modulus with the parameters describing the R-curve for the toughened ceramics. (a) $m_U = 5$; (b) $m_U = 10$; (c) $m_U = 15$. For each data point, the fracture toughness, K_L, of the corresponding base material is kept to be 5 MPa·m$^{1/2}$.

ceramics can be described well with Weibull distribution equation. Also, the Weibull modulus, i.e., the slope of the straight line between ln ln $1/(1 - P_f)$ and ln σ_f for each toughened ceramic is significantly higher than that of the corresponding un-toughened base material, indicating that improvement in the strength variability results from toughening. Similar conclusions were also obtained by analyzing the simulation results for other "imagined" toughened ceramics.

Fig. 3 shows the Weibull moduli, m_T, of toughened ceramics as functions of the prescribed values of K_H and λ, two key parameters which give a complete description for the R-curve characteristics. An interesting feature of these plots is that, for a fixed K_H-value, the $m_T - \lambda$ curve passes through a well-defined maximum and this maximum point shifts to the large λ side as K_H increases. These findings seem to say that a suitable combination of λ and K_H may yield a narrower strength distribution.

Following the analysis of Ramachandran and Shetty [20], K_H is the fracture toughness of the toughened ceramic and λ can be considered to be a rough measure of the size of the steady-state process zone formed directly behind the crack-tip during crack extension. Thus, the relations between m_T and R-curve parameters shown in Fig. 3 may be useful for the microstructural design of toughened ceramics aiming at improving the reliability. During the past years, various toughening mechanisms have been proposed and analyzed theoretically and experimentally [9, 10, 29, 30]. Several models have been established to predict the increment in fracture toughness of the toughened ceramics relative to the base material. Based on these models, the three parameters, K_H, K_L and λ included in Equation 4 may be estimated roughly for a given ceramic toughened with a certain mechanism and then, based on the simulation results shown in Fig. 3, the most effective toughening mechanism, which would make the toughened material exhibit a narrower strength distribution, may be determined by comparing the expected Weibull modulus.

Fig. 3 indicates that the Weibull modulus, m_T, of toughened ceramic is always larger than that of the un-toughened base material. This is consistent with the previous analysis [11–13]. However, no effort was devoted directly to give a possible explanation for the physical origin of the increased Weibull modulus in the previous studies. Therefore, a further discussion on this issue should be conducted.

As analyzed in Section 2, when the specimen of the un-toughened base material is loaded to fracture, the contact-induced surface cracks would extend stably from its initial size, $(c_0)_B$, to the critical size, $(c_c)_B$, due to the decreasing tendency of the residual stress intensity with increasing crack size. Theoretical calculation gives $(c_c)_B/(c_0)_B = 2.5$ (cf. Equation 12). Undoubtedly, for the toughened ceramic, the existence of rising R-curve behavior would make the surface cracks extend stably over a longer distance compared with those in the un-toughened base material. In other words, the ratio of c_c/c_0 for the toughened ceramics would be larger than that for the base material. This analysis was confirmed by our numerical simulation results which are partially shown in Fig. 4, where the c_c/c_0 ratio is plotted as functions of the prescribed values of K_H and λ for a fixed prescribed m_U-value of 10. Clearly, the c_c/c_0-values for all the toughened ceramics are larger than 2.5, implying that toughening significantly improves the crack tolerance.

By noting the similarity between the $c_c/c_0 - \lambda$ curves in Fig. 4 and the $m_T - \lambda$ curves in Fig. 3, it can be

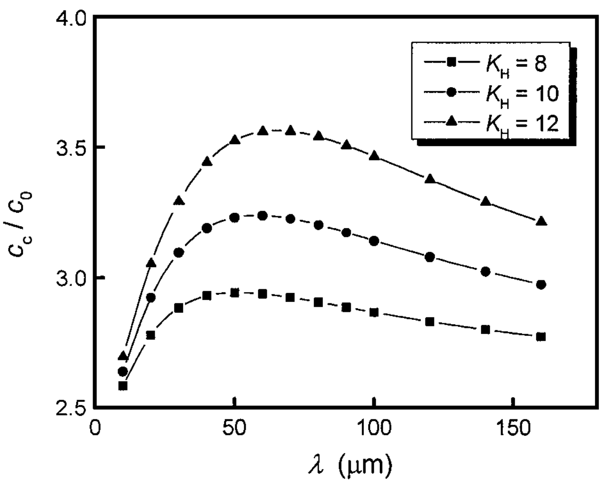

Figure 4 The ratio of c_c/c_0 as functions of the R-curve parameters for toughened ceramics. For each data point, the corresponding base material has a Weibull modulus $m_U = 10$.

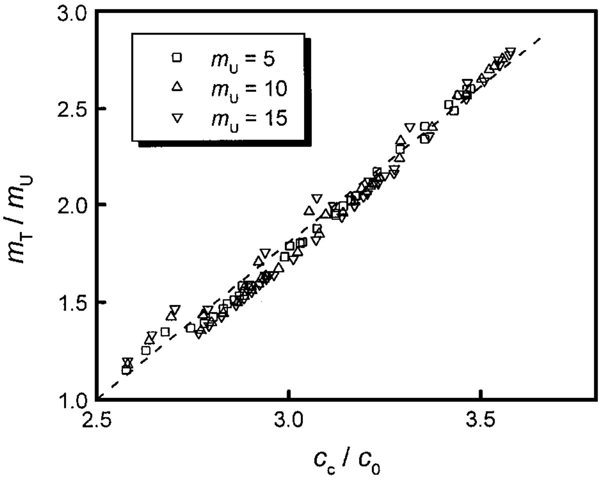

Figure 5 Variation of m_T/m_U with the relative crack tolerance c_c/c_0.

expected that, for the toughened ceramics, there may exist a strong correlation between the Weibull modulus and the relative crack tolerance c_c/c_0. In continuation of this idea, we plot the ratio of m_T/m_U, which may be considered as a measure of the improvement of the strength variability due to toughening, as a function of the ratio of c_c/c_0 in Fig. 5. It is of interest to note that all the data points fall along a straight line which passes through a characteristic point defined by $m_T = m_U$ and $c_c = 2.5\ c_0$. Thus, one can conclude that the increase in Weibull modulus of toughened ceramics is a consequence of the improvement of crack stability due to toughening.

5. Summary and conclusions

One of the main conclusions deduced from the above discussion is that the existence of rising R-curve behavior narrows the strength distribution, i.e., increases the Weibull modulus. This conclusion is consistent with those obtained previously by other authors [11–13]. However, the theoretical background for the present analysis is rather different with those used in the previous studies. Although an explicit solution for the change in Weibull modulus as a function of R-curve parameters cannot be derived in the present study due to the complicacy of the R-curve equation selected, the main improvement in analysis is that, compared with the previous studies, we have paid more attention to the microstructural effects on the increase in Weibull modulus. We related the Weibull modulus directly to the R-curve parameters which have reasonable physical meanings and, as a result, such a treatment makes it possible to evaluate the R-curve effect on strength variability for toughened ceramics with certain toughening mechanisms. Furthermore, a brief discussion on the physical origin of the increase in Weibull modulus, which was rarely mentioned in the previous studies, was also conducted in the present study. The increase in Weibull modulus was found to be related directly to the relative crack tolerance, i.e., the ratio of the initial crack size to the critical crack size. This suggests that the improvement in crack stability due to toughening is the main reason for the increased Weibull modulus.

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